# L' Hospital's Rule (general case)

1. Let $s \in \mathbb{R} \cup \{\pm\infty\}$

   assume $\lim_{x \to s} f(x) = 0 = \lim_{x \to s} g(x)$

   If $\lim_{x \to s} \frac{f'(x)}{g'(x)} = L \implies \lim \frac{f(x)}{g(x)} = L$

2. Same statement holds if $\lim_{x \to s} f(x) = \pm\infty = \lim_{x \to s} g(x)$

Last class:
- proved case 1 for $s \neq \pm\infty$
- outline for proof for $s = \pm\infty$

# Examples

① $\lim_{x\to 0_+} x \ln x = \lim_{x\to 0_+} \frac{\ln x}{1/x} \overset{②}{=} \lim_{x\to 0_+} \frac{1/x \cdot x^2}{-1/x^2 \cdot x^2}$

$x>0$ $\qquad \ln x$: $\log_e x$ in book

$= \lim_{x\to 0}(-x) = 0$

② $\lim_{x\to 0_+} x^x = \lim_{x\to 0_+} e^{x\ln x} = e^{\left(\lim_{x\to 0} x\ln x\right)} = e^0 = 1$

$x = e^{\ln x}$ for $x>0$

exponential function is continuous

Example 1

③ $\lim_{x \to \infty} \left(1 - \frac{1}{x}\right)^x = ? = \lim_{x\to\infty} e^{x \log(1 - 1/x)} = \boxed{e^{-1}}$

use same trick as before, using exp. function and logarithm

$$\left(1 - \frac{1}{x}\right)^x = e^{x \log\left(1 - \frac{1}{x}\right)}$$

enough to calculate

$$\lim_{x \to \infty} x \log\left(1 - \frac{1}{x}\right) = \lim_{x \to \infty} \frac{\log\left(1 - \frac{1}{x}\right)}{1/x}$$

$$\underset{\text{l'Hospital}}{=} \lim_{x \to \infty} \frac{\left(1 - \frac{1}{x}\right)^{-1} \cdot \cancel{x^{-2}}}{-\cancel{x^{-2}}}$$

$$= \lim_{x \to \infty} -\left(1 - \underset{\downarrow\, 0}{\frac{1}{x}}\right)^{-1} = -1$$

# Taylor's Theorem

Assume $f(x) = \sum_{k=0}^{\infty} a_k x^k$ for $|x| < R$

$R > 0$ (possibly $R = \infty$)

$\Rightarrow \quad f'(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$ ← also has radius of convergence $R$!

$\Rightarrow \quad f''(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$

can again differentiate term by term!

$\Rightarrow$ . . . . . repeat

$$\Rightarrow \quad \boxed{f^{(n)}(x) = \sum_{k=n}^{\infty} k(k-1)(k-2)\cdots(k-n)\, a_k x^{k-n}}$$

Can recover coefficients of power series from derivatives.

$$\boxed{f^{n}(0) = \frac{1}{n!} a_n}$$

<u>Def</u> $f$ defined on open interval $I$, $c \in I$

Assume $f^{(n)}(c)$ exists for all $n \geq 1$.

(1) Then the Taylor series for $f$ about $c$ is given by

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

(2) The $n$-th remainder $R_n(x)$ of $f$ is defined by

$$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(c)}{k!} (x-c)^k$$

**Remark**: Taylor series converges to $f(x)$

$\iff \quad R_n(x) \to 0$

## Taylor's Theorem

$f$ defined on $(a,b)$, $a < c < b$

assume $f^{(k)}(c)$ exists for $1 \le k \le n$

$\Rightarrow$ For all $x \neq c$ in $(a,b)$ there exists a $y$ between $x$ and $c$ such that

$$\boxed{R_n(x) = \frac{f^{(n)}(y)}{n!}(x-c)^n}$$

Proof. Let $M$ be such that

$$\boxed{M\frac{(x-c)^n}{n!} = \underbrace{f(x) - \sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k}_{=R_n(x)}} \quad (*)$$

<u>to show:</u> $\exists y$ between $x$ and $c$
s.t. $f^{(n)}(y) = M$.

Consider

$$\boxed{g(t) = \underbrace{\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(t-c)^k}_{=f(c) \text{ for } t=c} + \frac{M(t-c)^n}{n!} - \underbrace{f(t)}_{=f(c) \text{ for } t=c}} \quad (*)$$

observe: for $t=c$ we get $\boxed{g(c) = 0}$

by same argument: $\boxed{g^{(k)}(c) = 0}$ (evaluate $(*)$ to get $g^{(k)}(c) = f^{(k)}(c) - f^{(k)}(c)$)

$\boxed{g(x) = 0}$ ← follows from definition of $M$ $(*)$

$\Rightarrow \quad g(x) = 0 = g(c) \underset{\text{Rolle's Th.}}{\Rightarrow} \exists\, x_1$ between $x$ and $c$ s.t. $g'(x_1) = 0$

$g'(x_1) = 0 = g'(c) \Rightarrow \exists\, x_2$ " $x_1$ " " " $g''(x_2) = 0$

$g''(x_2) = 0 = g''(c) \Rightarrow \exists\, x_3$ " $x_2$ " " " $g'''(x_3) = 0$

repeat

$\Rightarrow \exists\, x_n$ between $x_{n-1}$ and $c$ s.t. $g^{(n)}(x_n) = 0$

pick $\boxed{y = x_n}$

$$g^{(n)}(y) = 0 = \underbrace{0 + M - f^{(n)}(y)}_{\text{take } n\text{-th derivatives on r.h.s. of } (*)}$$

$\Rightarrow$ claim

**Corollary** If all derivatives of $f$ exist for all $x \in (a,b)$ and $\exists\, C > 0$ s.t. $|f^{(n)}(x)| < C \quad \forall n,\ x \in (a,b)$

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0$$

and $f(x)$ given by Taylor series around $c \in (a,b)$.